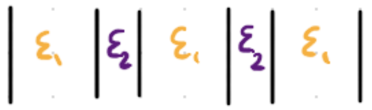
15) Photonic crystals 101

1. Definition

Photonic crystal = medium with a periodic distribution of ε and μ (1D, 2D, 3D)

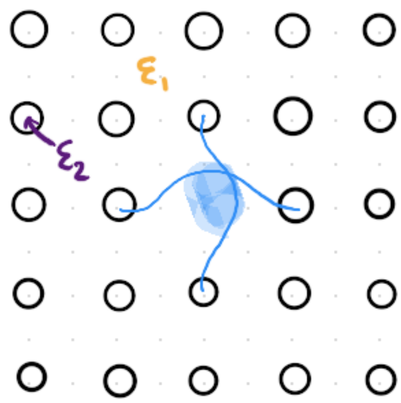
1D



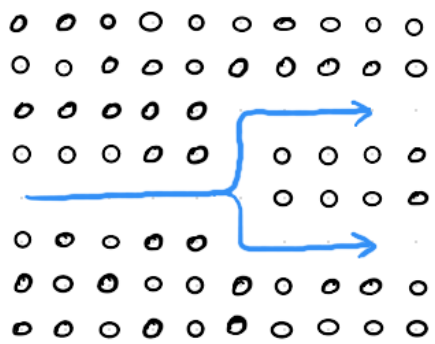
all dielectric Bragg mirrors

Coupled Resonator Optical Waveguides (CROWS)

2D

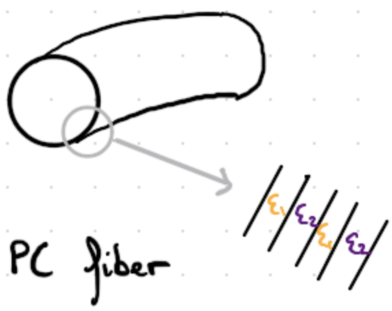


Photonic crystal defect cavity mode

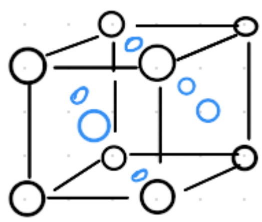


Photonic crystal devices

3D



PC fiber



fcc lattice

3-D photonic crystals

As in electronic crystals : band gaps, doping effects, anisotropy, topology.

2. Bravais lattice

Crystal = unit cell repeated infinitely

Bravais lattice + basis

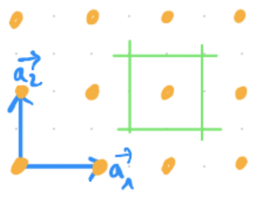
$\swarrow$ lattice vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$

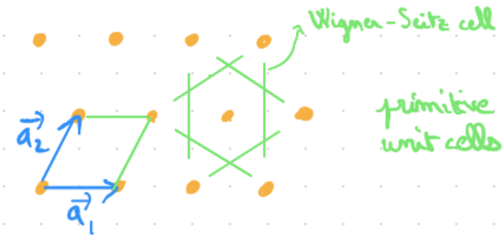
$\searrow$ what's inside the unit cell $\varepsilon(\vec{r}), \mu(\vec{r})$ $\vec{r} \in \mathcal{C}$

* Bravais lattice = ensemble of points spanned by

$$\vec{R} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + m_3 \vec{a}_3 \quad , m_i \in \mathbb{Z}$$

* 2D examples of Bravais lattices





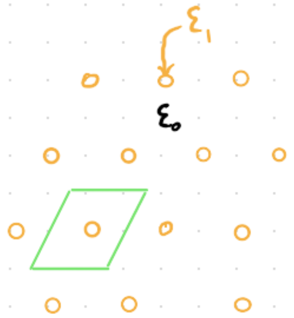
Square lattice

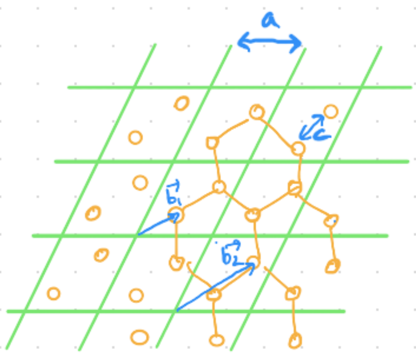
Triangular lattice

* $\begin{cases} \vec{a}_1 = a\,\vec{e}_x \\ \vec{a}_2 = a\,\vec{e}_y \end{cases}$

$\begin{cases} \vec{a}_1 = a\,\vec{e}_x \\ \vec{a}_2 = a\left(\frac{1}{2}\vec{e}_x + \frac{\sqrt{3}}{2}\vec{e}_y\right) \end{cases}$

2D examples of crystals with same lattice but $\neq$ basis





$\frac{a}{2} = c\frac{\sqrt{3}}{2}$

$\Downarrow$

$c = \frac{a}{\sqrt{3}}$

1 scatterer per unit cell: triangular

ε_1 at $\vec{b}_1 = \frac{1}{2}(\vec{a}_1 + \vec{a}_2)$

2 scatterers per unit cell: honeycomb

ε_1 at $\vec{b}_1 = c\left(\frac{\sqrt{3}}{2}\vec{e}_x + \frac{1}{2}\vec{e}_y\right)$

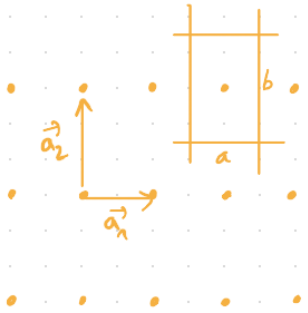
$\vec{b}_2 = c\left(\sqrt{3}\,\vec{e}_x + \vec{e}_y\right) = 2\vec{b}_1$

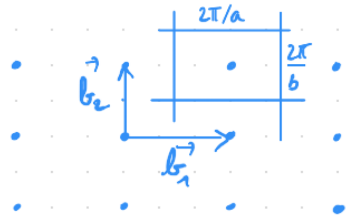
3. Reciprocal lattice

* Definition : Lattice generated from $\vec{b}_1, \vec{b}_2, \vec{b}_3$ defined by

$$\boxed{\vec{b}_i \cdot \vec{a}_j = 2\pi\,\delta_{ij}} \Rightarrow \vec{b}_1 = 2\pi \frac{\vec{a}_2 \times \vec{a}_3}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$ ← cell volume

* Examples



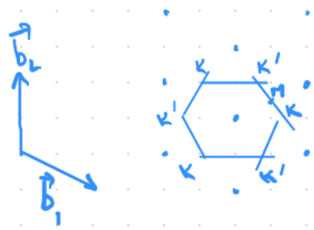


$$\begin{cases} \vec{a}_1 = a\,\vec{e}_x \\ \vec{a}_2 = b\,\vec{e}_y \end{cases}$$

$$\vec{b}_1 = 2\pi b \frac{\vec{e}_y \times \vec{e}_z}{ab} = \frac{2\pi}{a}\vec{e}_x$$

$$\vec{b}_2 = \frac{2\pi}{b}\vec{e}_y$$

Triangular lattice (exercice) : find $\vec{b}_1, \vec{b}_2$



* Reciprocal lattice vector : $\vec{G} = m_1\vec{b}_1 + m_2\vec{b}_2 + m_3\vec{b}_3$

Wigner Seitz cell of reciprocal space

* Useful to expand periodic functions

1D $\quad f(x + la) = f(x) \quad l \in \mathbb{Z}$

$$f(x) = \sum_m A_m e^{-j\frac{2\pi}{a}mx} \qquad \vec{G} = m\frac{2\pi}{a}\vec{e}_x = g\,\vec{e}_x \qquad \vec{x} = x\vec{e}_x$$

$$\vec{G}\cdot\vec{x} = gx$$

$$\Rightarrow f(x) = \sum_{\vec{G}} A_{\vec{G}}\, e^{-j\vec{G}\cdot\vec{x}}$$

How to get $A_{\vec{G}}$?

$$\frac{1}{a}\int_{cell} f(x)\, e^{+jg'x}\, dx = \sum_{\vec{G}} \frac{1}{a}\int_{cell} e^{-j(g-g')x} A_G\, dx = \sum_{\vec{G}} \delta_{GG'} A_{G'} = A_{G'}$$

3D
$$\boxed{f(\vec{r}) = \sum_{\vec{G}} A_{\vec{G}}\, e^{-j\vec{G}\cdot\vec{r}} \qquad A_{\vec{G}} = \frac{1}{V_{cell}}\int_{cell} f(\vec{r})\, e^{+j\vec{G}\cdot\vec{r}}\, d^3\vec{r}}$$

4. Maxwell's equations as a Hermitian eigenvalue problem

$$\begin{cases} \vec{B} = \mu(\vec{r})\vec{H} \\ \vec{D} = \varepsilon(\vec{r})\vec{E} \end{cases} \Rightarrow \begin{cases} \vec{\nabla}\times\vec{E} = -j\omega\mu(\vec{r})\vec{H} \\ \vec{\nabla}\times\vec{H} = j\omega\varepsilon(\vec{r})\vec{E} \end{cases}$$

$$\Rightarrow \begin{cases} \omega\vec{E} = -\frac{j}{\varepsilon}\vec{\nabla}\times\vec{H} \\ \omega\vec{H} = \frac{j}{\mu}\vec{\nabla}\times\vec{E} \end{cases} \qquad \omega\underbrace{\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}}_{\vec{f}} = \underbrace{\begin{pmatrix} 0 & -\frac{j}{\varepsilon}\vec{\nabla}\times \\ \frac{j}{\mu}\vec{\nabla}\times & 0 \end{pmatrix}}_{\text{not Hermitian?}}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$$

not Hermitian? $\Rightarrow \|\vec{f}\|^2$ not related to energy

$$\Rightarrow \omega\begin{pmatrix}\varepsilon\vec{E}\\ \mu\vec{H}\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -j\vec{\nabla}\times \\ j\vec{\nabla}\times & 0\end{pmatrix}}_{H_0}\underbrace{\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}}_{\vec{f}}$$

$$\Rightarrow \omega A\vec{f} = H_0\vec{f} \quad \text{with } A = \mathrm{diag}(\varepsilon,\mu) = \sqrt{A}\cdot\sqrt{A} \Rightarrow \omega\sqrt{A}\vec{f} = \sqrt{A}^{-1}H_0\vec{f}$$

$$\Rightarrow \omega\underbrace{\sqrt{A}\vec{f}}_{\vec{\Psi}} = \underbrace{\sqrt{A}^{-1}H_0\sqrt{A}^{-1}}_{H}\underbrace{\sqrt{A}\vec{f}}_{\vec{\Psi}} \Rightarrow \boxed{\omega\vec{\Psi} = H\vec{\Psi}}$$

with $\boxed{\vec{\Psi} = \begin{pmatrix}\sqrt{\varepsilon}\vec{E}\\ \sqrt{\mu}\vec{H}\end{pmatrix}}$ and $\boxed{H = \sqrt{A}^{-1}H_0\sqrt{A}^{-1} = \begin{pmatrix} 0 & -jc(\vec{r})\vec{\nabla}\times \\ jc(\vec{r})\vec{\nabla}\times & 0\end{pmatrix}}$

Hermitian

$\Rightarrow$ eigenvalues ω_m are real and modes form a complete orthogonal basis:

$$\int d^3\vec{r}\,\vec{\Psi}_m^{*}\cdot\vec{\Psi}_n = \delta_{mn} = \int d^3\vec{r}\,\left(\varepsilon(\vec{r})\vec{E}_m^{*}\cdot\vec{E}_n + \mu(\vec{r})\vec{H}_m^{*}\cdot\vec{H}_n\right)$$

Rk - Can be generalized to dispersive media

5. Bloch theorem

Consider an infinite periodic crystal $\Rightarrow \varepsilon(\vec{r}) = \varepsilon(\vec{r}+\vec{R})$ and $\mu(\vec{r}) = \mu(\vec{r}+\vec{R})$

$\Rightarrow \underline{H(\vec{r}) = H(\vec{r}+\vec{R})}$, with $\vec{R}$ any lattice vector.

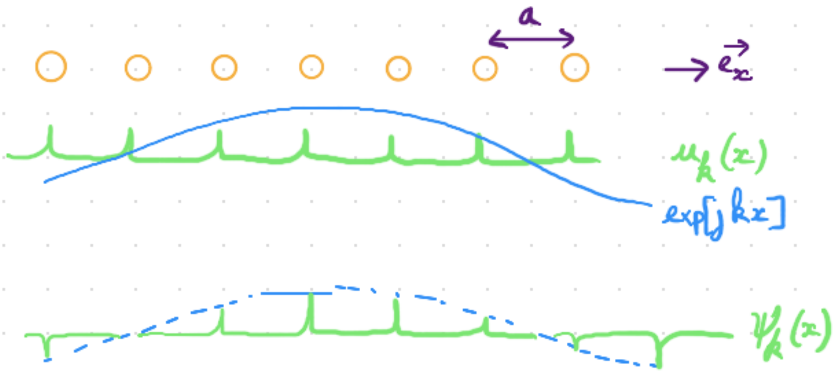
<u>Bloch theorem</u>: In a crystal, the solutions of $H\vec{\psi} = \omega\vec{\psi}$ called Bloch modes, are

$$\begin{cases} \vec{\psi}_k(\vec{r}) = e^{-j\vec{k}.\vec{r}}\,\vec{u}_k(\vec{r}) \\ \vec{u}_k(\vec{r}+\vec{R}) = \vec{u}_k(\vec{r}) \end{cases}$$

$\vec{k}$, Bloch wave number labels the solution

<u>Equivalent statement</u>: $\vec{\psi}_k(\vec{r}+\vec{R}) = e^{-j\vec{k}.\vec{R}}\,\vec{\psi}_k(\vec{r})$

1D example



<u>Proof in 1D</u> $H(x+a) = H(x)$. Let's define the translation operator T_a by $T_a : \vec{\psi}(x) \mapsto \vec{\psi}(x+a)$. The scalar product is $\int \psi^*(x)\psi(x)\,dx$ (L^2 norm).

T_a and H commute: $\forall\,\vec{\psi}$ $\quad T_a(H\vec{\psi}(x)) = H(x+a)\vec{\psi}(x+a) = H(x)\vec{\psi}(x+a)$

$= H(T_a\psi(x))$

$\Rightarrow$ They are diagonizable in same basis. Let $\vec{\psi}$ be a common eigenstate.

$$\begin{aligned} H\vec{\psi} &= \omega\vec{\psi} \\ T_a\vec{\psi} &= t_a\vec{\psi} \end{aligned} \quad \Rightarrow \quad \|T_a\vec{\psi}\| = |t_a|\,\|\vec{\psi}\| = \left(\int \psi^*(x+a)\psi(x+a)\,dx\right)$$

By change of variable, it's obvious that T is unitary $\Rightarrow \|T_a\vec{\psi}\| = \|\vec{\psi}\|$

$\Rightarrow |t_a| = 1 \Rightarrow t_a = e^{-jka} \Rightarrow \vec{\psi}(x+a) = e^{-jka}\vec{\psi}(x)$ [QED]

Proof in 3D : Similarly to 1D, one defines the translation operator:

$T_{\vec{R}}\, \vec{\psi}(\vec{r}) = \vec{\psi}(\vec{r}+\vec{R})$ $\quad [H, T_{\vec{R}}] = 0 \quad$ $T_{\vec{R}}$ unitary

$\Rightarrow$ Look for common eigenstates to H and $T_{\vec{R}}$

$$T_{\vec{R}}\, \vec{\psi}(\vec{r}) = t_{\vec{R}}\, \vec{\psi}(\vec{r})$$

Trick: Expand $\vec{\psi}(\vec{r})$ into plane waves : $\vec{\psi}(\vec{r}) = \int d\vec{q}\; C(\vec{q})\, e^{-j\vec{q}\cdot\vec{r}}$

$$\Rightarrow \int d\vec{q}\; C(q)\, e^{-j\vec{q}\cdot\vec{r}}\, e^{-j\vec{q}\cdot\vec{R}} = t_{\vec{R}} \int d\vec{q}\; C(\vec{q})\, e^{-j\vec{q}\cdot\vec{r}}$$

By taking $\int d\vec{q}\,'\, e^{j\vec{q}\,'\cdot\vec{r}} \times$ both sides $\Rightarrow$ $\cancel{C(\vec{q}\,')}\, e^{-j\vec{q}\,'\cdot\vec{R}} = t_{\vec{R}}\, \cancel{C(\vec{q}\,')}$

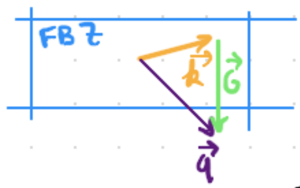
$$\Rightarrow \exp[-j\vec{q}\cdot\vec{R}] = t_{\vec{R}}$$

Write any $\vec{q}$ as $\boxed{\vec{q} = \vec{G} + \vec{k}}$

$t_{\vec{R}}$ does not depend on $\vec{G}$ choice since $e^{-j\vec{G}\cdot\vec{R}} = 1$

$\Rightarrow$ I can chose $\vec{G}$ such that $\vec{k}$ is in the Wigner Seitz primitive cell of the reciprocal lattice = First Brillouin zone

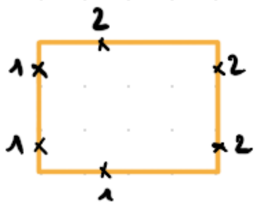
(example for a rectangular lattice :



)

$$\Rightarrow \vec{\psi}_k(\vec{r}+\vec{R}) = e^{-j\vec{k}\cdot\vec{R}}\, \vec{\psi}_k(\vec{r}) \quad \text{with } \vec{k} \in \text{FBZ}. \quad \text{QED}]$$

Corollary
$$\begin{cases} \vec{E}(\vec{r}+\vec{R}) = e^{-j\vec{k}\cdot\vec{R}}\, \vec{E}(\vec{r}) \\ \vec{H}(\vec{r}+\vec{R}) = e^{-j\vec{k}\cdot\vec{R}}\, \vec{H}(\vec{r}) \end{cases} \quad \text{since } \sqrt{\tfrac{\varepsilon}{\mu}(\vec{r}+\vec{R})} = \sqrt{\tfrac{\varepsilon}{\mu}(\vec{r})}$$

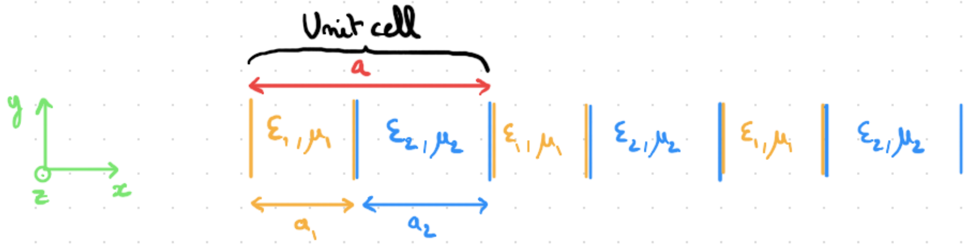


$\vec{E}_2 = e^{-j\vec{k}\cdot\vec{r}_{12}}\, \vec{E}_1$

Floquet PBC in FEM

6. Example in 1D

$\boxed{\text{1D Photonic crystal}}$: $\varepsilon(x+a)=\varepsilon(x)$, $\mu(x+a)=\mu(x)$



$n_1=\sqrt{\varepsilon_1\mu_1}$, $n_2=\sqrt{\varepsilon_2\mu_2}$, $c=1/\sqrt{\mu_0\varepsilon_0}$ $\quad \eta_0=\sqrt{\mu_0/\varepsilon_0}$

Dispersion relations: $\frac{\omega^2 n_1^2}{c^2}=\underbrace{k_1^2}_{\text{along } x}+\underbrace{k_y^2}_{\text{conserved}}$, $\frac{\omega^2 n_2^2}{c^2}=\underbrace{k_2^2}+\underbrace{k_y^2}$ $\qquad k_z^2=0$

2 polarizations: TE ($E_z, H_x, H_y \neq 0$) or TM ($H_z, E_x, E_y \neq 0$)

Transfer matrix method + Bloch theorem

TE case:

$$\begin{pmatrix} E_z(x=a_1) \\ \eta_0 H_y(x=a_1)\end{pmatrix} = T_1 \begin{pmatrix} E_z(x=0) \\ \eta_0 H_y(x=0)\end{pmatrix} \quad \text{with} \quad T_1=\begin{pmatrix} \cos k_1a_1 & j\sin(k_1a_1)\frac{\omega}{ck_1} \\ j\sin(k_1a_1)\frac{ck_1}{\omega} & \cos k_1a_1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E_z(x=a_2) \\ \eta_0 H_y(x=a_2)\end{pmatrix} = \underbrace{T_2T_1}_{T} \begin{pmatrix} E_z(x=0) \\ \eta_0 H_y(x=0)\end{pmatrix} \overset{\text{Bloch}}{=} e^{-jka} \begin{pmatrix} E_z(x=0) \\ \eta_0 H_y(x=0)\end{pmatrix}$$

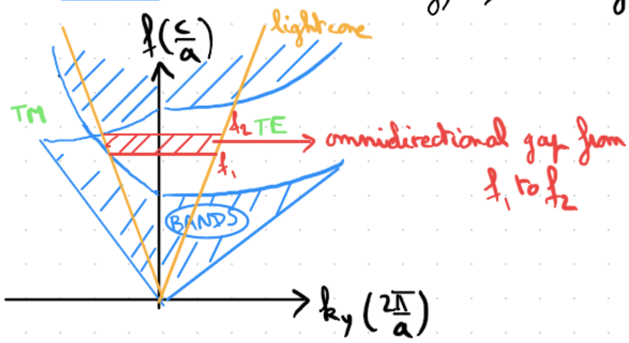
The eigenvalues of T give the k in Bloch theorem

$$\lambda_{1,2}=e^{\pm jka} \Rightarrow \operatorname{Tr} T = \lambda_1+\lambda_2 = 2\cos ka$$

$$\Rightarrow \boxed{\frac{1}{2}\operatorname{Tr} T = \cos ka = \cos k_1a_1 \cos k_2a_2 - \frac{1}{2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\sin k_1a_1 \sin k_2a_2}$$

RHS $<1 \Rightarrow k\in\mathbb{R} \Rightarrow$ band
$\quad\;\; >1 \Rightarrow k\in j\mathbb{R} \Rightarrow$ gap

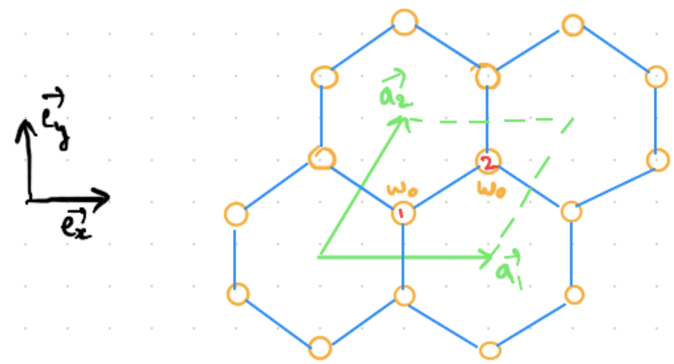
exercice: TM case (duality) $\Rightarrow$ all angle TE/TM mirrors



quantity	unit
lattice	a
freq	$\frac{c}{a}$
k	$2\pi/a$
λ	a

7. Example in 2D

$\boxed{\text{Photonic graphene}}$ Triangular lattice with 2 resonators $\Rightarrow$ honeycomb



nearest neighbor coupling K

$\vec{a_1} = a\,\vec{e_x}$

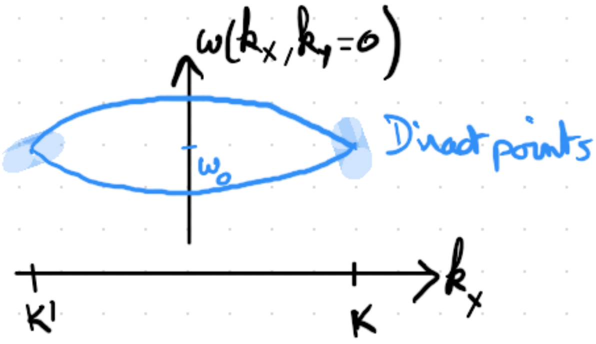
$\vec{a_2} = \frac{a}{2}\vec{e_x} + a\frac{\sqrt{3}}{2}\vec{e_y}$

Coupled mode theory + bloch

$$\begin{cases} -j\mathring{A}_1 = \omega_0 A_1 + K A_2 + K e^{+j\vec{k}.\vec{a_2}} A_2 + K e^{+j\vec{k}.\vec{a_1}} A_2 \\ -j\mathring{A}_2 = \omega_0 A_2 + K A_1 + K e^{j\vec{k}.\vec{a_2}} A_1 + K e^{-j\vec{k}.\vec{a_1}} A_1 \end{cases}$$

$$\Rightarrow -j\partial_t \vec{\Psi}_k = \hat{H}(k)\,\vec{\Psi}_k = \omega\,\vec{\Psi}_k \qquad \hat{H}(k) = \begin{pmatrix} \omega_0 & K(1+e^{j\vec{k}.\vec{a_2}} + e^{j\vec{k}.\vec{a_1}}) \\ cc & \omega_0 \end{pmatrix}$$

$\Rightarrow \omega_\pm(k) \Rightarrow$ 2 bands ! (Plot = exercise)



Not gapped.